

The evolution of density perturbations in $f(R)$ gravity

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We give a rigorous and mathematically well defined presentation of the Covariant and Gauge Invariant theory of scalar perturbations of a Friedmann-Lemaître-Robertson-Walker universe for Fourth Order Gravity, where the matter is described by a perfect fluid with a barotropic equation of state. The general perturbations equations are applied to a simple background solution of R^n gravity. We obtain exact solutions of the perturbations equations for scales much bigger than the Hubble radius. These solutions have a number of interesting features. In particular, we find that for all values of n there is always a growing mode for the density contrast, even if the universe undergoes an accelerated expansion. Such a behaviour does not occur in standard General Relativity, where as soon as Dark Energy dominates, the density contrast experiences an unrelenting decay. This peculiarity is sufficiently novel to warrant further investigation of fourth order gravity models.

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I. INTRODUCTION

In the last few years the idea of a geometrical origin for Dark Energy (DE) i.e. the connection between DE and a non-standard behavior of gravitation on cosmological scales has attracted a considerable amount of interest.

Higher order gravity, and in particular fourth order gravity, has been widely studied in the case of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric using a number of different techniques (see for example [1, 2, 3, 4, 5, 6, 7]). Recently a general approach was developed to analyze the phase space of the fourth order cosmologies [8, 9, 10], providing for the first time a way of obtaining exact solutions together with their stability and a general idea of the qualitative behavior of these cosmological models.

The phase space analysis shows that for FLRW models there exist classes of fourth order theories which admit a transient decelerated expansion phase, followed by one with an accelerated expansion rate (see also [7] for a different approach). The first (Friedmann-like) phase provides a setting during which structure formation can take place, followed by a smooth transition to a DE-like era which drives the cosmological acceleration. However, in order to determine if this is the case, we need to develop a theory of cosmological perturbations for higher order gravity.

The aim of this paper is to give a rigorous and mathematically well defined theory which describes the evolution scalar perturbations of FLRW models in fourth order gravity, which can be used to investigate this issue in detail (see [11, 12, 13, 14] for other recent contributions to this area).

In order to achieve this goal a perturbation formalism needs to be chosen that is best suited for this task. One possible choice is the Bardeen metric based approach [15] which guarantees the gauge invariance of the results. However this approach has the drawback of introducing variables which only have a clear physical meaning in certain gauges [16]. Although this is not a big problem in the context of General Relativity (GR), this is not necessarily true in the case of higher order gravity and consequently could lead to a miss-interpretation of the results.

In what follows we will use, instead, the covariant and gauge invariant approach developed for GR in [16, 17, 18, 19, 20, 21] which has the advantage of using perturbation variables with a clear geometrical and physical interpretation. Furthermore, we use a specific recasting of the field equations that will make the development of the cosmological perturbation theory even more transparent.

The outline of the paper is as follows. In Section II we give the general set up of the equations. In Section III we introduce the 1+3 covariant approach and apply it to the field equations presented in Section I. In Section IV we give the general evolution equations for the 1+3 quantities and linearize them around a FLRW spacetime. In Section V we introduce the perturbation variables and their propagation equations giving also their harmonic decomposition.

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In Section VI we apply these equation to the case of R^n -gravity with a barotropic perfect fluid matter source and find exact solution in long wavelength limit. Finally in Section VII we present our conclusions.

Unless otherwise specified, natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper, Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $- , + , + , +$ signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = W^a{}_{bd,c} - W^a{}_{bc,d} + W^e{}_{bd}W^a{}_{ce} - W^f{}_{bc}W^a{}_{df} , \quad (1)$$

where the $W^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices), defined by

$$W^a{}_{bd} = \frac{1}{2}g^{ae} (g_{be,d} + g_{ed,b} - g_{bd,e}) . \quad (2)$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd}R_{acbd} . \quad (3)$$

Finally the Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{A} = \int dx^4 \sqrt{-g} \left[\frac{1}{2}R + L_m \right] . \quad (4)$$

II. GENERAL EQUATIONS FOR FOURTH ORDER GRAVITY.

The most general action for a fourth order theory of gravity is given by

$$\mathcal{A} = \int d^4x \sqrt{-g} [\Lambda + c_0 R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \mathcal{L}_m] , \quad (5)$$

where we have used the Gauss Bonnet theorem [22] and \mathcal{L}_m represents the matter contribution. In situations where the metric has a high degree of symmetry, this action can be further simplified. In particular, in the homogeneous and isotropic case the (5) can be expressed as

$$\mathcal{A} = \int d^4x \sqrt{-g} [f(R) + \mathcal{L}_m] . \quad (6)$$

Varying the action with respect to the metric gives the generalization of the Einstein equations:

$$f' G_{ab} = f' \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = T_{ab}^m + \frac{1}{2} g_{ab} (R - Rf') + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f' , \quad (7)$$

where $f = f(R)$, $f' = \frac{df(R)}{dr}$, and $T_{\mu\nu}^M = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}$ represents the stress energy tensor of standard matter.

These equations reduce to the standard Einstein field equations when $f(R) = R$. It is crucial for our purposes to be able to write (7) in the form

$$G_{ab} = \tilde{T}_{ab}^m + T_{ab}^R = T_{ab}^{tot} , \quad (8)$$

where $\tilde{T}_{ab}^m = \frac{T_{ab}^m}{f'}$ and

$$T_{ab}^R = \frac{1}{f'} \left[\frac{1}{2} g_{ab} (R - Rf') + \nabla_b \nabla_a f - g_{ab} \nabla_c \nabla^c f \right] , \quad (9)$$

represent two effective “fluids”: the *curvature “fluid”* (associated with T_{ab}^R) and the *effective matter “fluid”* (associated with \tilde{T}_{ab}^m). This step is important because it allows us to treat fourth order gravity as standard Einstein gravity in the presence of two “effective” fluids. This means that once the effective thermodynamics of these fluids has been studied, we can apply the covariant gauge invariant approach in the standard way.

The conservation properties of these effective fluids are given by the Bianchi identities $T_{ab}^{tot};{}^b = 0$. When applied to the total stress energy tensor, these identities reveal that if standard matter is conserved, the total fluid is also conserved

even though the curvature fluid may in general possess off-diagonal terms [8, 23, 24]. In other words, no matter how complicated the effective stress energy tensor T_{ab}^{tot} is, it will always be divergence free if $T_{ab}^{m;b} = 0$. When applied to the single effective tensors, the Bianchi identities read

$$\tilde{T}_{ab}^{M;b} = \frac{T_{ab}^{m;b}}{f'} - \frac{f''}{f'^2} T_{ab}^m R^b, \quad (10)$$

$$T_{ab}^{R;b} = \frac{f''}{f'^2} \tilde{T}_{ab}^M R^b, \quad (11)$$

with the last expression being a consequence of total energy-momentum conservation. It follows that the individual effective fluids are not conserved but exchange energy and momentum.

It is worth noting here that even if the energy-momentum tensor associated with the effective matter source is not conserved, standard matter still follows the usual conservation equations $T_{ab}^{m;b} = 0$. It is also important to stress that the fluids with T_{ab}^R and \tilde{T}_{ab}^m defined above are *effective* and consequently can admit features that one would normally consider un-physical for a standard matter field. This means that all the thermodynamical quantities associated with the curvature defined below should be considered *effective* and not bounded by the usual constraints associated with matter fields. It is important to understand that this does not compromise any of the thermodynamical properties of standard matter represented by the Lagrangian \mathcal{L}_m .

III. COVARIANT DECOMPOSITION OF HIGHER ORDER GRAVITY

In this section we will describe the general covariant decomposition fourth order gravity. This procedure will take place in two steps. The first one is to develop the kinematics of the spacetime and the second one is to study the thermodynamics of the effective fluids defined in the previous section.

A. Preliminaries

The starting point for any analysis using the covariant approach is the choice of a suitable frame i.e. the 4-velocity u_a of an observer in spacetime.

This choice changes the structure of the equations and can simplify the calculations in the same way a choice of coordinates makes life easier in classical mechanics. Even if the covariance of the theory guarantees that all velocity fields are equivalent, a number of natural choices for u_a exist; they are the *energy frame* u_a^E , which is defined to be a timelike eigenvector of the stress energy tensor T_{ab} , the *particle frame* u_a^N that is derived from the particle flux vector N_a and the *entropy frame* u_a^S defined by the entropy flux vector S_a . These frames have the advantage of inducing important simplifications to the equations (e.g. in the energy frame the total energy flux is always zero [16]).

The vectors u_a^E, u_a^N, u_a^S always exist in the case of a perfect fluid and coincide, defining a unique hydrodynamical 4-velocity for the flow [16]. In the case of more than one perfect fluid, one can in principle define these three frames for each component as well as for the total fluid and then choose the most convenient frame to work in.

In our specific case, equation (8) allows us to define two “effective” fluids, but their structure does not necessarily make any of the three frames defined above a suitable choice. This because both the curvature fluid and the effective matter do not necessarily satisfy the Weak Energy Condition (WEC) $T_{ab}V^aV^b \geq 0, V^aV_a \leq 0$ [25]. This relation, which for homogeneous and isotropic spacetimes corresponds to the requirement that the energy density is positive, is the key hypothesis which allows the timelike vectors u_a^E, u_a^N, u_a^S to exist and is, in general, a very reasonable assumption. In our case, however, the violation of this condition means that none of the energy entropy or particle frames are, in general suitable choices of frame.

However, an alternative frame choice follows from the fact that whatever the behavior of the effective fluid is, standard matter is still thermodynamically well defined and consequently the stress energy tensor \tilde{T}_{ab} satisfies the standard energy conditions. It follows that a natural choice of frame the one of those associated with standard matter ($u_a = u_a^m$), assumed to be a barotropic perfect fluid with equation of state $p = w\mu$. [34]. This choice is also motivated by the fact that the real observers are attached to galaxies and these galaxies follow the standard matter geodesics. Consequently this frame choice is the one which can be best motivated from a physical point of view.

B. Kinematics

Once the frame has been chosen the derivation of the kinematical quantities can be obtained in a standard way [26]. The derivative along the matter fluid flow lines is defined by $\dot{X} = u_a \nabla^a X$. The projection tensor into the tangent 3-spaces orthogonal to the flow vector is:

$$h_{ab} \equiv g_{ab} + u_a u_b \Rightarrow h^a_b h^b_c = h^a_c, \quad h_{ab} u^b = 0, \quad (12)$$

and the tensor $\nabla_b u_a$ can be expanded as

$$\nabla_b u_a = \tilde{\nabla}_b u_a - a_a u_b, \quad \tilde{\nabla}_b u_a = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \quad (13)$$

where $\tilde{\nabla}_a = h^b_a \nabla_b$ is the spatially totally projected covariant derivative operator orthogonal to u^a . This relation allows us to define the key kinematic quantities of the cosmological model: the expansion Θ , the shear σ_{ab} , the vorticity ω_{ab} and the acceleration $a_a = \dot{u}_a$.

In the following, angle brackets applied to a vector denote the projection of this vector on the tangent 3-spaces

$$V_{\langle a \rangle} = h_a^b V_b. \quad (14)$$

Instead when applied to a tensor they denote the projected, symmetric and trace free part of this object

$$W_{\langle ab \rangle} = [h_{(a}^c h_{b)}^d - \frac{1}{3} h^{cd} h_{ab}] W_{cd}. \quad (15)$$

Finally the spatial curl of a variable is

$$({}_{\cdot} X)^{ab} = \eta^{cd \langle a} \tilde{\nabla}_c X^{b \rangle}_d \quad (16)$$

where $\epsilon_{abc} = u^d \eta_{abcd}$ is the spatial volume.

The general propagation equations for these kinematic variables, for any spacetime corresponds to the so called *1+3 covariant equations* [26] and are given in Appendix C.

C. Effective total energy-momentum tensors

The choice of the frame also allows us to obtain an irreducible decomposition of the stress energy momentum tensor. In a general frame and for a general tensor T_{ab} one obtains:

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{\langle a} u_{b \rangle} + \pi_{ab}, \quad (17)$$

where μ and p are the energy density and isotropic pressure, q_a is the energy flux ($q_a = q_{\langle a \rangle}$) and π_{ab} is the anisotropic pressure ($\pi_{ab} = \pi_{\langle ab \rangle}$).

This decomposition can be applied to our effective energy momentum tensors. Relative to u_a^m we obtain

$$\mu^{\text{tot}} = T_{ab}^{\text{tot}} u^a u^b = \tilde{\mu}^m + \mu^R, \quad p^{\text{tot}} = \frac{1}{3} T_{ab}^{\text{tot}} h^{ab} = \tilde{p}^m + p^R, \quad (18)$$

$$q_a^{\text{tot}} = -T_{bc}^{\text{tot}} h_a^b u^c = \tilde{q}_a^m + q_a^R, \quad \pi_{ab}^{\text{tot}} = T_{cd}^{\text{tot}} h_{\langle a}^c h_{b \rangle}^d = \tilde{\pi}_{ab}^m + \pi_{ab}^R, \quad (19)$$

with

$$\tilde{\mu}^m = \frac{\mu^m}{f'}, \quad \tilde{p}^m = \frac{p^m}{f'}, \quad \tilde{q}_a^m = \frac{q_a^m}{f'}, \quad \tilde{\pi}_{ab}^m = \frac{\pi_{ab}^m}{f'}. \quad (20)$$

Since we assume that standard matter is a perfect fluid, q_a^m and π_{ab}^m are zero, so that the last two quantities above also vanish.

The effective thermodynamical quantities for the curvature “fluid” are

$$\mu^R = \frac{1}{f'} \left[\frac{1}{2} (Rf' - f) - \Theta f'' \dot{R} + f'' \tilde{\nabla}^2 R + f'' \dot{u}_b \tilde{\nabla} R \right], \quad (21)$$

$$p^R = \frac{1}{f'} \left[\frac{1}{2} (f - Rf') + f'' \ddot{R} + 3f''' \dot{R}^2 + \frac{2}{3} \Theta f'' \dot{R} - \frac{2}{3} f'' \tilde{\nabla}^2 R + \right. \\ \left. - \frac{2}{3} f''' \tilde{\nabla}^a R \tilde{\nabla}_a R - \frac{1}{3} f'' \dot{u}_b \tilde{\nabla} R \right], \quad (22)$$

$$q_a^R = -\frac{1}{f'} \left[f''' \dot{R} \tilde{\nabla}_a R + f'' \tilde{\nabla}_a \dot{R} - \frac{1}{3} f'' \tilde{\nabla}_a R \right], \quad (23)$$

$$\pi_{ab}^R = \frac{1}{f'} \left[f'' \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b \rangle} R + f''' \tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b \rangle} R + \sigma_{ab} \dot{R} \right]. \quad (24)$$

The twice contracted Bianchi Identities lead to evolution equations for μ^m , μ^R , q_a^R :

$$\dot{\mu}^m = -\Theta (\mu^m + p^m), \quad (25)$$

$$\dot{\mu}^R + \tilde{\nabla}^a q_a^R = -\Theta (\mu^R + p^R) - 2 (\dot{u}^a q_a^R) - (\sigma^{ab} \pi_{ba}^R) + \mu^m \frac{f'' \dot{R}}{f'^2}, \quad (26)$$

$$\dot{q}_{\langle a}^R + \tilde{\nabla}_a p^R + \tilde{\nabla}^b \pi_{ab}^R = -\frac{4}{3} \Theta q_a^R - \sigma_a^b q_b^R - (\mu^R + p^R) \dot{u}_a - \dot{u}^b \pi_{ab}^R - \eta_a^{bc} \omega_b q_c^R + \mu^m \frac{f'' \tilde{\nabla}_a R}{f'^2} \text{leq : cons3}, \quad (27)$$

and a relation connecting the acceleration \dot{u}_a to μ^m and p^m follows from momentum conservation of standard matter:

$$\tilde{\nabla}^a p^m = -(\mu^m + p^m) \dot{u}^a. \quad (28)$$

Note that, as we have seen in the previous section the curvature fluid and the effective matter exchange energy and momentum. The decomposed interaction terms in Equations (26) and (25) are given by $\mu^m \frac{f'' \tilde{\nabla}_a R}{f'^2}$ and $\mu^m \frac{f'' \dot{R}}{f'^2}$.

IV. PROPAGATION AND CONSTRAINT EQUATIONS

A. Nonlinear equations

We are now ready to write the full nonlinear 1+3 equations for higher order gravity. Substituting the quantities given above into the equations given in the Appendix A, we obtain the following results:

Expansion propagation (generalized Raychaudhuri equation):

$$\dot{\Theta} + \frac{1}{3} \Theta^2 + \sigma_{ab} \sigma^{ab} - 2\omega_a \omega^a - \tilde{\nabla}^a \dot{u}_a + \dot{u}_a \dot{u}^a + \frac{1}{2} (\tilde{\mu}^m + 3\tilde{p}^m) = -\frac{1}{2} (\mu^R + 3p^R), \quad (29)$$

Vorticity propagation:

$$\dot{\omega}_{\langle a} + \frac{2}{3} \Theta \omega_a + \frac{1}{2} \text{curl} \dot{u}_a - \sigma_{ab} \omega^b = 0, \quad (30)$$

Shear propagation:

$$\dot{\sigma}_{\langle ab \rangle} + \frac{2}{3} \Theta \sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a} \dot{u}_{b \rangle} + \sigma_{c \langle a} \sigma_{b \rangle}^c + \omega_{\langle a} \omega_{b \rangle} - \dot{u}_{\langle a} \dot{u}_{b \rangle} = \frac{1}{2} \pi_{ab}^R, \quad (31)$$

Gravito-electric propagation:

$$\dot{E}_{\langle ab \rangle} + \Theta E_{ab} - \text{curl} H_{ab} + \frac{1}{2} (\tilde{\mu}^m + \tilde{p}^m) \sigma_{ab} - 2\dot{u}^c \varepsilon_{cd \langle a} H_{b \rangle}^d - 3\sigma_{c \langle a} E_{b \rangle}^c + \omega^c \varepsilon_{cd \langle a} E_{b \rangle}^d \\ = -\frac{1}{2} (\mu^R + p^R) \sigma_{ab} - \frac{1}{2} \dot{\pi}_{\langle ab \rangle}^R - \frac{1}{2} \tilde{\nabla}_{\langle a} q_{b \rangle}^R - \frac{1}{6} \Theta \pi_{ab}^R - \frac{1}{2} \sigma^c_{\langle a} \pi_{b \rangle c}^R - \frac{1}{2} \omega^c \varepsilon_{c \langle a} \pi_{b \rangle d}^R, \quad (32)$$

Gravito-magnetic propagation:

$$\dot{H}_{\langle ab \rangle} + \Theta H_{ab} + \text{curl} E_{ab} - 3\sigma_{c \langle a} H_{b \rangle}^c + \omega^c \varepsilon_{cd \langle a} H_{b \rangle}^d + 2\dot{u}^c \varepsilon_{cd \langle a} E_{b \rangle}^d \\ = \frac{1}{2} \text{curl} \pi_{ab}^R - \frac{3}{2} \omega_{\langle a} q_{b \rangle}^R + \frac{1}{2} \sigma^c_{\langle a} \varepsilon_{b \rangle c}^d q_d^R, \quad (33)$$

Vorticity constraint:

$$\tilde{\nabla}^a \omega_a - \dot{u}^a \omega_a = 0, \quad (34)$$

Shear constraint:

$$\tilde{\nabla}^b \sigma_{ab} - \text{curl} \omega_a - \frac{2}{3} \tilde{\nabla}_a \Theta + 2[\omega, \dot{u}]_a = -q_a^R, \quad (35)$$

Gravito-magnetic constraint:

$$\text{curl} \sigma_{ab} + \tilde{\nabla}_{\langle a} \omega_{b \rangle} - H_{ab} + 2\dot{u}_{\langle a} \omega_{b \rangle} = 0, \quad (36)$$

Gravito-electric divergence:

$$\tilde{\nabla}^b E_{ab} - \frac{1}{3} \tilde{\nabla}_a \tilde{\mu}^m - [\sigma, H]_a + 3H_{ab} \omega^b = \frac{1}{2} \sigma_a^b q_b^R - \frac{3}{2} [\omega, q^R]_a - \frac{1}{2} \tilde{\nabla}^b \pi_{ab}^R + \frac{1}{3} \tilde{\nabla}_a \mu^R - \frac{1}{3} \Theta q_a^R, \quad (37)$$

Gravito-magnetic divergence:

$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m) \omega_a + [\sigma, E]_a - 3E_{ab} \omega^b = -\frac{1}{2} \text{curl} q_a^R + (\mu^R + p^R) \omega_a - \frac{1}{2} [\sigma, \pi^R]_a - \frac{1}{2} \pi_{ab}^R \omega^b. \quad (38)$$

The standard GR equations are obtained by setting $f(R) = R$ which corresponds to setting all the right hand sides of these equations to zero. Together with Eqs. (25)–(28), these equations govern the dynamics of the matter and gravitational fields in fourth order gravity. As we will see the new source terms in the propagation and constraint equations will modify the evolution of the perturbations in a non-trivial way.

B. Linearized equations

In the previous section we derived the exact nonlinear equations that govern the exact gravitational dynamics of fourth order gravity relative to observers comoving with standard matter. These equations are fully covariant and hold for any spacetime. Consequently, we can linearize these equation around any chosen background, avoiding the need for choosing coordinates and dealing directly with physically well defined quantities, rather than metric components [35]. These features, which are desirable in the GR case, become essential for the correct understanding of the evolution of perturbations in fourth order gravity as well as in other kinds of alternative gravity theories [27].

In what follows we will choose a Friedmann-Lemaître-Robertson-Walker (FLRW) metric as our background. We make this choice for a number of different reasons. First of all the possibility of writing a general fourth order lagrangian as a simple function of the Ricci scalar is surely possible for this metric. Secondly, because most of the work in GR perturbation theory has been performed for this background it makes a comparison of behavior of GR and fourth order gravity more straightforward.

The Friedmann background is characterized by the vanishing of all inhomogeneous and anisotropic quantities q_a^R, π_{ab}^R and defines the order of the quantities appearing in the 1+3 equations and the linearization procedure. In particular, the quantities that are zero in the background are considered first-order of in the linearization scheme. In addition, the Stuart & Walker lemma ensures that since these quantities vanish in the background, they are automatically gauge-invariant [28].

The cosmological equations for the background read:

$$\Theta^2 = 3\tilde{\mu}^m + 3\mu^R - \frac{\tilde{R}}{2}, \quad (39)$$

$$\dot{\Theta} + \frac{1}{3} \Theta^2 + \frac{1}{2} (\tilde{\mu}^m + 3\tilde{p}^m) + \frac{1}{2} (\mu^R + 3p^R) = 0, \quad (40)$$

$$\dot{\mu}^m + \Theta (\mu^m + p^m) = 0, \quad (41)$$

where μ^R and p^R are the zero order energy density and pressure of the curvature fluid, \tilde{R} is the 3-Ricci scalar and $\tilde{R} = 6K/S^2$ with the spatial curvature index $K = 0, \pm 1$.

Linearization of the exact propagation and constraint equations about this background then leads to the system:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 - \tilde{\nabla}^a A_a + \frac{1}{2}(\tilde{\mu}^m + 3\tilde{p}^m) = -\frac{1}{2}(\mu^R + 3p^R), \quad (42)$$

$$\dot{\omega}_a + 2H\omega_a + \frac{1}{2}\text{curl} A_a = 0, \quad (43)$$

$$\dot{\sigma}_{ab} + 2H\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a} A_{b\rangle} = -q_a^R, \quad (44)$$

$$\begin{aligned} \dot{E}_{ab} + 3HE_{ab} - \text{curl} H_{ab} + \frac{1}{2}(\tilde{\mu}^m + \tilde{p}^m)\sigma_{ab} \\ = -\frac{1}{2}(\mu^R + p^R)\sigma_{ab} - \frac{1}{2}\dot{\pi}_{\langle ab\rangle}^R - \frac{1}{2}\tilde{\nabla}_{\langle a} q_{b\rangle}^R - \frac{1}{6}\Theta\pi_{ab}^R, \end{aligned} \quad (45)$$

$$\dot{H}_{ab} + 3HH_{ab} + \text{curl} E_{ab} = \frac{1}{2}\text{curl} \pi_{ab}^R, \quad (46)$$

$$\tilde{\nabla}^b \sigma_{ab} - \text{curl} \omega_a - \frac{2}{3}\tilde{\nabla}_a \Theta = -q_a^R, \quad (47)$$

$$\text{curl} \sigma_{ab} + \tilde{\nabla}_{\langle a} \omega_{b\rangle} - H_{ab} = 0, \quad (48)$$

$$\tilde{\nabla}^b E_{ab} - \frac{1}{3}\tilde{\nabla}_a \tilde{\mu}^m = -\frac{1}{2}\tilde{\nabla}^b \pi_{ab}^R + \frac{1}{3}\tilde{\nabla}_a \mu^R - \frac{1}{3}\Theta q_a^R, \quad (49)$$

$$\tilde{\nabla}^b H_{ab} - (\tilde{\mu}^m + \tilde{p}^m)\omega_a = -\frac{1}{2}\text{curl} q_a^R + (\mu^R + p^R)\omega_a, \quad (50)$$

$$\tilde{\nabla}^a \omega_a = 0, \quad (51)$$

together with the linearized conservation equations

$$\dot{\mu}^m = -\Theta(\mu^m + p^m), \quad (52)$$

$$\tilde{\nabla}^a p^m = -(\mu^m + p^m)\dot{u}^a, \quad (53)$$

$$\dot{\mu}^R + \tilde{\nabla}^a q_a^R = -\Theta(\mu^R + p^R) + \mu^m \frac{f'' \dot{R}}{f'^2}, \quad (54)$$

$$\dot{q}_{\langle a}^R + \tilde{\nabla}_a p^R + \tilde{\nabla}^b \pi_{ab}^R = -\frac{4}{3}\Theta q_a^R - (\mu^R + p^R)\dot{u}_a + \mu^m \frac{f'' \tilde{\nabla}_a R}{f'^2}, \quad (55)$$

obtained from (25)–(28). Note that at first order the equation of the vorticity (51) is homogeneous i.e. the evolution of the vorticity is decoupled. This will be important in the next section when we will derive the perturbations equations. These equations provide the basis for a covariant and gauge-invariant description of perturbations of $f(R)$ theories of gravity.

V. DYNAMICS OF SCALAR PERTURBATIONS

A. Perturbation Equations

We are now ready to analyze the evolution of the density perturbations on a FLRW background. The quantities appearing in the linearized equations given in the previous section can be decomposed in scalar vector and tensor components, i.e.

$$V_a = \bar{V}_a + \hat{V}_a = \eta^{abc}\tilde{\nabla}_b \bar{V}_c + \tilde{\nabla}^a \hat{v}, \quad \text{where} \quad \tilde{\nabla}^a \bar{V}_a = 0, \quad \eta^{abc}\tilde{\nabla}_b \hat{V}_c = 0, \quad (56)$$

and

$$W_{ab} = \bar{W}_{ab} + \hat{W}_{ab} + W_{ab}^* = \bar{W}_{ab} + \tilde{\nabla}_a \bar{W}_b + \tilde{\nabla}_a \tilde{\nabla}_b W^*, \quad (57)$$

where

$$\tilde{\nabla}^a \bar{W}_{ab} = 0, \quad (\hat{W})_{ab} = 0, \quad (W^*)_{ab} = 0, \quad (58)$$

and both of these decompositions are unique. Note that here we define scalars, vectors or tensors as quantities that transform like scalars, solenoidal vectors or symmetric tensors, or are obtained from them using the h_{ab} or $\tilde{\nabla}_a$ operators [18].

In linear regime and in homogeneous and isotropic backgrounds these different components do not interact with each other. In the following we will focus only on the evolution of scalar perturbations because they are directly related with density fluctuations. This can be done simply discarding the non scalar quantities in the equation above i.e. setting

$$V_a = \tilde{\nabla}_a V, \quad W_{ab} = \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b\rangle} W. \quad (59)$$

The identities in Appendix C, the vorticity constraint equation (34) and the gravito-magnetic constraint equation (36) then show that

$$\text{curl } V_a = 0 = \text{curl } W_{ab}, \quad \tilde{\nabla}^b W_{ab} = \frac{2}{3} \tilde{\nabla}^2 (\tilde{\nabla}_a W), \quad \omega_a = 0 = H_{ab}, \quad (60)$$

as in standard General Relativity.

In order to derive the equations governing density perturbations in the general case, we define the density and expansion gradients

$$\mathcal{D}_a^m = \frac{S}{\mu^m} \tilde{\nabla}_a \mu^m, \quad Z_a = S \tilde{\nabla}_a \Theta, \quad C_a = S \tilde{\nabla}_a \tilde{R}, \quad (61)$$

and the (dimensionless) gradients describing inhomogeneity in the Ricci scalar:

$$\mathcal{R}_a = S \tilde{\nabla}_a R, \quad \mathfrak{R}_a = S \tilde{\nabla}_a \dot{R}. \quad (62)$$

Another important quantity in the treatment of the evolution of the density perturbations is the Newtonian potential (defined through the divergence of the electric part of the Weyl tensor (37) [18]).

$$\begin{aligned} \Phi_a^N &= S^2 \mu^{tot} \mathcal{D}_a^{tot} \\ &= \frac{2S^2 \mu \Theta}{2\Theta f' + 3\dot{R} f''} \mathcal{D}_a^m + \frac{3\dot{R} f''}{2(2\Theta f' + 3\dot{R} f'')} C_a - \frac{2S^2 f'' \Theta^2}{2\Theta f' + 3\dot{R} f''} \mathfrak{R}_a \\ &\quad + \frac{S^2 \left[f'' (f - 2\mu + 2\dot{R} \Theta f'') - 2\dot{R} \Theta f' f^{(3)} \right] \Theta}{f' (2\Theta f' + 3\dot{R} f'')} \mathcal{R}_a \end{aligned} \quad (63)$$

where \mathcal{D}_a^{tot} represents the total energy density fluctuation.

Using equations (42)–(50), equations (25)–(28), the identities in Appendix C, assuming matter to be a barotropic perfect fluid with barotropic factor $w = p^m/\mu^m$ and that the vorticity is zero [36], we obtain the following system of evolution equations for the above variables:

$$\dot{\mathcal{D}}_a^m = w \Theta \mathcal{D}_a^m - (1+w) Z_a, \quad (64)$$

$$\begin{aligned} \dot{Z}_a &= \left(\frac{\dot{R} f''}{f'} - \frac{2\Theta}{3} \right) Z_a + \left[\frac{3(w-1)(3w+2)}{6(w+1)} \frac{\mu}{f'} + \frac{2w\Theta^2 + 3w(\mu^R + 3p^R)}{6(w+1)} \right] \mathcal{D}_a^m + \frac{\Theta f''}{f'} \mathfrak{R}_a \\ &\quad + \left[\frac{1}{2} - \frac{f'' K}{f' S^2} - \frac{1}{2} \frac{f f''}{f' f'} - \frac{f'' \mu}{f' f'} + \dot{R} \Theta \left(\frac{f''}{f'} \right)^2 + \dot{R} \Theta \frac{f^{(3)}}{f'} \right] \mathcal{R}_a - \frac{w}{w+1} \tilde{\nabla}^2 \mathcal{D}_a^m - \frac{f''}{f'} \tilde{\nabla}^2 \mathcal{R}_a, \end{aligned} \quad (65)$$

$$\dot{\mathcal{R}}_a = \mathfrak{R}_a - \frac{w}{w+1} \dot{R} \mathcal{D}_a^m, \quad (66)$$

$$\begin{aligned} \dot{\mathfrak{R}}_a &= - \left(\Theta + 2\dot{R} \frac{f^{(3)}}{f''} \right) \mathfrak{R}_a - \dot{R} Z_a - \left[\frac{(3w-1)\mu}{3f''} + 3 \frac{w}{w+1} (p^R + \mu^R) \frac{f'}{f''} + \frac{w}{3(w+1)} \dot{R} \left(\Theta - 3\dot{R} \frac{f^{(3)}}{f''} \right) \right] \mathcal{D}_a^m \\ &\quad \left[3 \frac{K}{S^2} - \left(\frac{1}{3} \frac{f'}{f''} + \frac{f^{(4)}}{f'} \dot{R}^2 + \Theta \frac{f^{(3)}}{f'} \dot{R} - \frac{2}{9} \Theta^2 + \frac{1}{3} (\mu^R + 3p^R) + \dot{R} \frac{f^{(3)}}{f''} - \frac{1}{6} \frac{f}{f'} + \frac{1}{2} (w+1) \frac{\mu}{f'} - \frac{1}{3} \dot{R} \Theta \frac{f''}{f'} \right) \right] \mathcal{R}_a \\ &\quad + \tilde{\nabla}^2 \mathcal{R}_a, \end{aligned} \quad (67)$$

together with the constraint

$$\frac{C_a}{S^2} + \left(\frac{4}{3} \Theta + \frac{2\dot{R} f''}{f'} \right) Z_a - 2 \frac{\mu}{f'} \mathcal{D}_a^m + \left[2\dot{R} \Theta \frac{f^{(3)}}{f'} - \frac{f''}{f'} \left(f - 2\mu + 2\dot{R} \Theta f'' + 2 \frac{K}{S^2} \right) \right] \mathcal{R}_a + \frac{2\Theta f''}{f'} \mathfrak{R}_a - \frac{2f''}{f'} \tilde{\nabla}^2 \mathcal{R}_a = 0. \quad (68)$$

The propagation equation for the variable C is

$$\begin{aligned}
\dot{C}_a = & K^2 \left[\frac{36f''\mathcal{R}_a}{S^2(2\Theta f' + 3\dot{R}f'')} - \frac{36f'\mathcal{D}_a^m}{S^2(2\Theta f' + 3\dot{R}f'')} \right] + K \left\{ \frac{6f'}{S^2(2\Theta f' + 3\dot{R}f'')} C_a \right. \\
& + \mathcal{D}_a^m \left(\frac{16\omega\Theta}{3(\omega+1)} - \frac{4f'\Theta^2 - 12f'\mu^R}{2\Theta f' + 3\dot{R}f''} \right) - \frac{12f''}{2\Theta f' + 3\dot{R}f''} \tilde{\nabla}^2 \mathcal{R}_a + \left(\frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} + 2\frac{f''}{f'} \right) \mathfrak{R}_a \\
& + \left. \left[-\frac{2S^2(\Theta f'' - 3\dot{R}f^{(3)})}{3f'} - \frac{12\dot{R}\Theta f' f^{(3)} - 2f''(3f - 2(\Theta^2 - 3\mu^R)f' + 6\dot{R}\Theta f'')}{(2\Theta f' + 3\dot{R}f'')f'} \right] \mathcal{R}_a \right\} \\
& + \tilde{\nabla}^2 \left[\frac{4\omega S^2\Theta}{3(\omega+1)} \mathcal{D}_a^m + \frac{2S^2 f''}{f'} \mathfrak{R}_a - \frac{2S^2(\Theta f'' - 3\dot{R}f^{(3)})}{3f'} \mathcal{R}_a \right], \tag{69}
\end{aligned}$$

this equation, which is redundant, will be used in Section VI to substitute (66) because of its specific form in the long wavelength limit [29].

B. Scalar Variables

The variables we have defined above describe the general evolution of the density perturbations and the other scalars on a FLRW background. The phenomenon of the clustering of matter is traditionally described, however, considering only the scalar part of these variables. This can be easily done using the *local* decomposition [18]

$$S\tilde{\nabla}^a X_a = X_{ab} = \frac{1}{3}h_{ab}X + \Sigma_{ab}^X + X_{[ab]} \quad \text{where} \quad \Sigma_{ab}^X = X_{(ab)} - \frac{1}{3}h_{ab}X. \tag{70}$$

so that the operator $\tilde{\nabla}_a$ applied to the (61) and (62) extracts the scalar part of the perturbation variables. In this way we can define the scalar quantities

$$\Delta_m = S\tilde{\nabla}^a \mathcal{D}_a^m, \quad Z = S\tilde{\nabla}^a Z_a, \quad C = S\tilde{\nabla}^a C_a, \quad \mathcal{R} = S\tilde{\nabla}^a \mathcal{R}_a, \quad \mathfrak{R} = S\tilde{\nabla}^a \mathfrak{R}_a, \quad \Phi^N = S\tilde{\nabla}^a \Phi_a^N. \tag{71}$$

which will characterize the evolution of the spherically symmetric part of the gradients (61-62). The evolution equations for the first four of these variables are

$$\dot{\Delta}_m = w\Theta\Delta_m - (1+w)Z, \tag{72}$$

$$\begin{aligned}
\dot{Z} = & \left(\frac{\dot{R}f''}{f'} - \frac{2\Theta}{3} \right) Z + \left[\frac{3(w-1)(3w+2)\mu}{6(w+1)f'} + \frac{2w\Theta^2 + 3w(\mu^R + 3p^R)}{6(w+1)} \right] \Delta_m + \frac{\Theta f''}{f'} \mathfrak{R} \\
& + \left[\frac{1}{2} - \frac{1}{2} \frac{f}{f'} \frac{f''}{f'} - \frac{f''\mu}{f'f'} + \dot{R}\Theta \left(\frac{f''}{f'} \right)^2 + \dot{R}\Theta \frac{f^{(3)}}{f'} \right] \mathcal{R} - \frac{w}{w+1} \tilde{\nabla}^2 \Delta_m - \frac{f''}{f'} \tilde{\nabla}^2 \mathcal{R}, \tag{73}
\end{aligned}$$

$$\dot{\mathfrak{R}} = \mathfrak{R} - \frac{w}{w+1} \dot{R} \Delta_m, \tag{74}$$

$$\begin{aligned}
\dot{\mathfrak{R}} = & - \left(\Theta + 2\dot{R} \frac{f^{(3)}}{f''} \right) \mathfrak{R} - \dot{R}Z - \left[\frac{(3w-1)\mu}{3f''} + 3\frac{w}{w+1}(p^R + \mu^R) \frac{f'}{f''} + \frac{w}{3(w+1)} \dot{R} \left(\Theta - 3\dot{R} \frac{f^{(3)}}{f''} \right) \right] \Delta_m \\
& + \left[2\frac{K}{S^2} - \left(\frac{1}{3} \frac{f'}{f''} + \frac{f^{(4)}}{f'} \dot{R}^2 + \Theta \frac{f^{(3)}}{f'} \dot{R} - \frac{2}{9} \Theta^2 + \frac{1}{3}(\mu^R + 3p^R) + \dot{R} \frac{f^{(3)}}{f''} - \frac{1}{6} \frac{f}{f'} + \frac{1}{2}(w+1) \frac{\mu}{f'} - \frac{1}{3} \dot{R} \Theta \frac{f''}{f'} \right) \right] \mathcal{R} \\
& + \tilde{\nabla}^2 \mathcal{R}, \tag{75}
\end{aligned}$$

$$\begin{aligned}
\dot{C} = & K^2 \left[\frac{36f''\mathcal{R}}{S^2(2\Theta f' + 3\dot{R}f'')} - \frac{36f'\Delta}{S^2(2\Theta f' + 3\dot{R}f'')} \right] + K \left[\frac{6f'}{S^2(2\Theta f' + 3\dot{R}f'')} C + \Delta \left(\frac{4\omega\Theta}{\omega + 1} - \frac{4f'\Theta^2 - 12f'\mu^R}{2\Theta f' + 3\dot{R}f''} \right) \right. \\
& \left. - \frac{12f''}{2\Theta f' + 3\dot{R}f''} \tilde{\nabla}^2 \mathcal{R} + \frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} \mathfrak{R} + \frac{12\dot{R}\Theta f' f^{(3)} - 2f''(3f - 2(\Theta^2 - 3\mu^R)f' + 6\dot{R}\Theta f'')}{(2\Theta f' + 3\dot{R}f'')f'} \mathcal{R} \right] \\
& + \tilde{\nabla}^2 \left[\frac{4\omega S^2 \Theta}{3(\omega + 1)} \Delta + \frac{2S^2 f''}{f'} \mathfrak{R} - \frac{2S^2(\Theta f'' - 3\dot{R}f^{(3)})}{3f'} \mathcal{R} \right], \tag{76}
\end{aligned}$$

together with the constraint

$$\frac{C}{S^2} + \left(\frac{4}{3}\Theta + \frac{2\dot{R}f''}{f'} \right) Z - 2\frac{\mu}{f'} \Delta_m + \left[2\dot{R}\Theta \frac{f^{(3)}}{f'} - \frac{f''}{f'} (f - 2\mu + 2\dot{R}\Theta f'') \right] \mathcal{R} + \frac{2\Theta f''}{f'} \mathfrak{R} - \frac{2f''}{f'} \tilde{\nabla}^2 \mathcal{R} = 0. \tag{77}$$

In standard GR, only the first two equations and the last one are present and the density perturbations are governed by a second-order equation for Δ^m whose independent solutions are adiabatic growing and decaying modes. The presence of fourth order corrections introduces important changes to this picture. In fact, in this case the evolution of the density perturbations is described by a closed *fourth* order differential equation which can be obtained from the above first order equations. This follows clearly from our two effective fluids interpretation.

C. Harmonic analysis

The system (72)-(75) is a system of four partial differential equations which is far too complicated to be solved directly. For this reason, following a standard procedure we perform a harmonic decomposition. This allows one to reduce equations (72)-(75) to ordinary differential equations which are somewhat easier to solve.

In the covariant approach the harmonic decomposition is performed using the trace-free symmetric tensor eigenfunctions of the spatial the Laplace-Beltrami operator defined by [16]:

$$\tilde{\nabla}^2 Q = -\frac{k^2}{a^2} Q, \tag{78}$$

where $k = 2\pi S/\lambda$ is the wavenumber and $\dot{Q} = 0$. Using these harmonics we can expand every first order quantity in the equations above [37],

$$X(t, \mathbf{x}) = \sum X^{(k)}(t) Q^{(k)}(\mathbf{x}) \tag{79}$$

where \sum stands for both a summation over a discrete index or an integration over a continuous one.

Developing (71) in terms of Q , (72)-(68) reduce to

$$\dot{\Delta}_m^{(k)} = w\Theta\Delta_m^{(k)} - (1+w)Z^{(k)}, \quad (80)$$

$$\begin{aligned} \dot{Z}^{(k)} = & \left(\frac{\dot{R}f''}{f'} - \frac{2\Theta}{3} \right) Z^{(k)} + \left[\frac{3(w-1)(3w+2)\mu}{6(w+1)f'} + \frac{2w\Theta^2 + 3w(\mu^R + 3p^R)}{6(w+1)} - \frac{w}{w+1} \frac{k}{S^2} \right] \Delta_m^{(k)} + \frac{\Theta f''}{f'} \mathfrak{R}^{(k)} \\ & + \left[\frac{1}{2} - \frac{f''}{f'} \frac{k}{S^2} - \frac{1}{2} \frac{f}{f'} \frac{f''}{f'} - \frac{f''}{f'} \frac{\mu}{f'} + \dot{R}\Theta \left(\frac{f''}{f'} \right)^2 + \dot{R}\Theta \frac{f^{(3)}}{f'} \right] \mathcal{R}^{(k)}, \end{aligned} \quad (81)$$

$$\dot{\mathcal{R}}^{(k)} = \mathfrak{R}^{(k)} - \frac{w}{w+1} \dot{R} \Delta_m^{(k)}, \quad (82)$$

$$\begin{aligned} \dot{\mathfrak{R}}^{(k)} = & - \left(\Theta + 2\dot{R} \frac{f^{(3)}}{f''} \right) \mathfrak{R}^{(k)} - \dot{R}Z^{(k)} - \left[\frac{(3w-1)\mu}{3f''} + 3\frac{w}{w+1}(p^R + \mu^R) \frac{f'}{f''} + \frac{w}{3(w+1)} \dot{R} \left(\Theta - 3\dot{R} \frac{f^{(3)}}{f''} \right) \right] \Delta_m^{(k)} \\ & + \left[\frac{k}{S^2} + 2\frac{K}{S^2} - \left(\frac{1}{3} \frac{f'}{f''} + \frac{f^{(4)}}{f'} \dot{R}^2 + \Theta \frac{f^{(3)}}{f'} \dot{R} - \frac{2}{9} \Theta^2 \right. \right. \\ & \left. \left. + \frac{1}{3}(\mu^R + 3p^R) + \dot{R} \frac{f^{(3)}}{f''} - \frac{1}{6} \frac{f}{f'} + \frac{1}{2}(w+1) \frac{\mu}{f'} - \frac{1}{3} \dot{R}\Theta \frac{f''}{f'} \right) \right] \mathcal{R}^{(k)}, \end{aligned} \quad (83)$$

$$\begin{aligned} \dot{C}^{(k)} = & K^2 \left[\frac{36f''\mathcal{R}^{(k)}}{S^2(2\Theta f' + 3\dot{R}f'')} - \frac{36f'\Delta^{(k)}}{S^2(2\Theta f' + 3\dot{R}f'')} - \frac{12f''}{S^2(2\Theta f' + 3\dot{R}f'')} \mathcal{R}^{(k)} \right] \\ & + K \left[\frac{6f'}{S^2(2\Theta f' + 3\dot{R}f'')} C^{(k)} + \left(\frac{4\omega\Theta}{\omega+1} - \frac{4f'\Theta^2 - 12f'\mu^R}{2\Theta f' + 3\dot{R}f''} \right) \Delta^{(k)} \right. \\ & \left. + \frac{12\Theta f''}{2\Theta f' + 3\dot{R}f''} \mathfrak{R}^{(k)} + \frac{12\dot{R}\Theta f' f^{(3)} - 2f''(3f - 2(\Theta^2 - 3\mu^R)f' + 6\dot{R}\Theta f'')}{(2\Theta f' + 3\dot{R}f'')f'} \mathcal{R}^{(k)} \right] \\ & + \frac{k}{S^2} \left[\frac{4\omega S^2 \Theta}{3(\omega+1)} \Delta^{(k)} + \frac{2S^2 f''}{f'} \mathfrak{R}^{(k)} - \frac{2S^2(\Theta f'' - 3\dot{R}f^{(3)})}{3f'} \mathcal{R}^{(k)} \right], \end{aligned} \quad (84)$$

$$0 = \frac{C^{(k)}}{S^2} + \left(\frac{4}{3}\Theta + \frac{2\dot{R}f''}{f'} \right) Z^{(k)} - 2\frac{\mu}{f'} \Delta_m^{(k)} + \left[2\dot{R}\Theta \frac{f^{(3)}}{f'} - \frac{f''}{f'} (f - 2\mu + 2\dot{R}\Theta f'') - 2\frac{f''}{f'} \frac{k}{S^2} \right] \mathcal{R}^{(k)} + \frac{2\Theta f''}{f'} \mathfrak{R}^{(k)}. \quad (85)$$

Finally it is useful to write equations (80-83) as a pair of second order equation. In this way the GR limit is more transparent when the written in this form. They are:

$$\begin{aligned} \ddot{\Delta}^{(k)} - & \left[\left(\omega - \frac{2}{3} \right) \Theta + \frac{\dot{R}f''}{f'} \right] \dot{\Delta}^{(k)} - \left[\omega k^2 - \omega(3p^R + \mu^R) - \frac{2\omega\dot{R}\Theta f''}{f'} - \frac{(3\omega^2 - 1)\mu}{f'} \right] \Delta^{(k)} \\ = & \frac{1}{2}(w+1) \left[2\frac{k^2}{S^2} f'' + (f - 2\mu + 2\dot{R}\Theta f'') \frac{f''}{f'^2} - 2\dot{R}\Theta \frac{f^{(3)}}{f'} \right] \mathcal{R}^{(k)} - \frac{(w+1)\Theta f''}{f'} \dot{\mathcal{R}}^{(k)} \end{aligned} \quad (86)$$

$$\begin{aligned} f''\ddot{\mathcal{R}}^{(k)} + & (\Theta f'' + 2\dot{R}f^{(3)}) \dot{\mathcal{R}}^{(k)} - \left[\frac{k^2}{S^2} f'' + 2\frac{K}{S^2} f'' + \frac{2}{9}\Theta^2 f'' - (w+1) \frac{\mu}{2f'} f'' - \frac{1}{6}(\mu^R + 3p^R) f'' \right. \\ & \left. - \frac{f'}{3} + \frac{f}{6f'} f'' + \dot{R}\Theta \frac{f''^2}{6f'} - \dot{R}f^{(3)} - \Theta f^{(3)} \dot{R} - f^{(4)} \dot{R}^2 \right] \mathcal{R}^{(k)} = - \left[\frac{1}{3}(3w-1)\mu \right. \\ & \left. + \frac{w}{1+w} (f^{(3)} \dot{R}^2 + (p^R + \mu^R) f' + \frac{7}{3} \dot{R}\Theta f'' + \ddot{R}f'') \right] \Delta^{(k)} - \frac{(w-1)\dot{R}f''}{w+1} \dot{\Delta}^{(k)} \end{aligned} \quad (87)$$

In the GR limit we have $f = R$, so the above equations reduce to

$$\begin{aligned} \ddot{\Delta}^{(k)} - & (\omega - \frac{2}{3}) \Theta \dot{\Delta}^{(k)} - [\omega k^2 - (\frac{1}{2} + \omega - \frac{3}{2}w^2) \mu] \Delta^{(k)} = 0, \\ \mathcal{R} = & (3\omega - 1) \mu \Delta^{(k)}. \end{aligned} \quad (88)$$

The second of these equations is just the spatial Laplacian of the trace of the Einstein Field equations $R = 3p - \mu$.

VI. EXAMPLE: R^n -GRAVITY

Let us now apply the equations derived in the above sections to the simplest example of fourth order theory of gravity: R^n -gravity. In this theory $f(R) = \chi R^n$ and the action reads

$$\mathcal{A} = \int d^4x \sqrt{-g} [\chi R^n + L_M] , \quad (89)$$

where χ a the coupling constant with suitable dimensions and $\chi = 1$ for $n = 1$.

If $R \neq 0$ the field equations for this theory read

$$G_{ab} = \chi^{-1} \frac{\tilde{T}_{ab}^M}{nR^{n-1}} + T_{ab}^R \quad (90)$$

where

$$\tilde{T}_{ab}^M = \chi^{-1} \frac{T_{ab}^M}{nR^{n-1}} , \quad (91)$$

$$T_{ab}^R = (n-1) \left\{ -\frac{R}{2n} g_{ab} + \left[\frac{R^{;cd}}{R} + (n-2) \frac{R^{;c} R^{;d}}{R^2} \right] (g_{ca} g_{db} - g_{cd} g_{ab}) \right\} . \quad (92)$$

The FLRW dynamics of this model has been investigated via a complete phase space analysis in [8]. This analysis shows that for specific intervals of the parameter n there is a set of initial conditions with non zero measure for which the cosmic histories include a transient decelerated phase which evolves towards an accelerated expansion one. This first phase was argued to be suitable for the structure formation to take place.

In what follows we will analyze the evolution of the scalar perturbations during this phase in the long wavelength limit. In this approximation the wavenumber k is considered to be so small that the wavelength $\lambda = 2\pi S/k$ associated with it is much larger than the Hubble radius. Equation (78) then implies that all the Laplacians can be neglected and the spatial dependence of the perturbation variables can be factored out. It is also well known [29] that in this limit and in spatially flat ($K = 0$) backgrounds the (84) reduces to $\dot{C} = 0$ i.e. the variable C is conserved so that the number of perturbations equations can be reduced to three.

Let us now set the background to be the transient solution

$$S = S_0 t^{\frac{2n}{3(1+w)}} , \quad k = 0 , \quad \mu = \mu_0 t^{-2n} \quad (93)$$

of [8]. The expansion, the Ricci scalar, the curvature fluid pressure, the curvature fluid energy density and the effective matter energy density take the form:

$$\Theta = \frac{2n}{t(\omega + 1)} , \quad (94)$$

$$R = \frac{4n[4n - 3(\omega + 1)]}{3t^2(\omega + 1)^2} , \quad (95)$$

$$\mu^R = \frac{2(n-1)[2n(3\omega + 5) - 3(\omega + 1)]}{3t^2(\omega + 1)^2} , \quad (96)$$

$$p^R = \frac{2(n-1)[n(6\omega^2 + 8\omega - 2) - 3\omega(\omega + 1)]}{3t^2(\omega + 1)^2} , \quad (97)$$

$$\mu = \left(\frac{3}{4}\right)^{1-n} n\chi \left(\frac{n(4n - 3(\omega + 1))}{t^2(\omega + 1)^2}\right)^{n-1} \frac{4n^2 - 2(n-1)[2n(3\omega + 5) - 3(\omega + 1)]}{3(\omega + 1)^2 t^2} . \quad (98)$$

Substituting in the equations given above and passing to the long wavelength limit we obtain

$$\begin{aligned} \dot{\Delta}_m = & \left[-\frac{2n}{w+1} - \frac{6(n-1)n}{n+3(n-1)w-3} + 1 \right] \frac{\Delta_m}{t} - \frac{3(w+1)^2}{4a_0^2[n+3(n-1)w-3]} C_0 t^{1-\frac{4n}{3(w+1)}} \\ & + \frac{3(n-1)(w+1)^2[n(6w+8)-15(w+1)]}{4[n+3(n-1)w-3][4n-3(w+1)]} \mathcal{R} t - \frac{9(n-1)(w+1)^3 t^2}{4[n+3(n-1)w-3][4n-3(w+1)]} \mathfrak{R} t^2, \end{aligned} \quad (99)$$

$$\dot{\mathcal{R}} = \mathfrak{R} + \frac{8nw(4n-3(w+1))}{3(w+1)^3} \frac{\Delta_m}{t^3}, \quad (100)$$

$$\begin{aligned} \dot{\mathfrak{R}} = & \frac{2n(4n-3w-3)}{(w+1)(n+3(n-1)w-3)} \frac{C_0}{a_0^2} t^{-\frac{4n}{3(w+1)}-2} + 2 \left(\frac{3n(n-1)}{n+3(n-1)w-3} - \frac{n}{w+1} + 2n-4 \right) \frac{\mathfrak{R}}{t} \\ & + 2 \left(-\frac{9n(n-2)(n-1)}{n+3(n-1)w-3} - 2n^2 + 7n + \frac{3n^2(9n-26)+57}{9(w+1)(n-1)} + \frac{8n^2(n-2)}{9(w+1)^2(n-1)} - 6 \right) \frac{\mathcal{R}}{t^2} \\ & + \frac{16n(4n+3(n-1)w-3)(4n-3(w+1))((9w(w+1)+8)n^2-(3w(9w+8)+13)n+3(w+1)(6w+1))}{27(n-1)(w+1)^4(n+3(n-1)w-3)} t^4 \Delta_m, \end{aligned} \quad (101)$$

where C_0 is the conserved value for the quantity C . The evolution of density perturbations can then be decoupled via the third order equation

$$(n-1)\ddot{\Delta}_m - (n-1) \left(\frac{4n\omega}{\omega+1} - 5 \right) \frac{\dot{\Delta}_m}{t} + \mathcal{D}_1(n, \omega) \frac{\dot{\Delta}_m}{t^2} + \mathcal{D}_2(n, \omega) \frac{\Delta_m}{t^3} + \mathcal{D}_3(n, \omega) C_0 t^{-\frac{4n}{3(\omega+1)}-1} = 0 \quad (102)$$

where

$$\mathcal{D}_1(n) = -\frac{2(-9(2(n-1)n+1)\omega^2 + 6n(n(4n-7)+1)\omega + 18\omega + n(4n(8n-19)+33)+9)}{9(\omega+1)^2} \quad (103)$$

$$\mathcal{D}_2(n) = \frac{2((2n-1)\omega-1)(4n-3(\omega+1))(3(\omega+1)+n(-9\omega+n(6\omega+8)-13))}{9(\omega+1)^3} \quad (104)$$

$$\mathcal{D}_3(n) = -\frac{n(21\omega-6n(\omega+2)+31)-18(\omega+1)}{6a_0^2} \quad (105)$$

This equation admits the general solution

$$\Delta_m = K_1 t^{\frac{2n\omega}{\omega+1}-1} + K_2 t^{\alpha_+} + K_3 t^{\alpha_-} - K_4 \frac{C_0}{a_0^2} t^{2-\frac{4n}{3(\omega+1)}}, \quad (106)$$

where

$$\alpha_{\pm} = -\frac{1}{2} + \frac{n\omega}{\omega+1} \pm \frac{\sqrt{(n-1)(4(3\omega+8)^2 n^3 - 4(3\omega(18\omega+55)+152)n^2 + 3(\omega+1)(87\omega+139)n - 81(\omega+1))}}{6(n-1)(\omega+1)^2} \quad (107)$$

$$K_4 = \frac{9(\omega+1)^3(18(\omega+1)+n(-21\omega+6n(\omega+2)-31))}{8(n(6\omega+4)-9(\omega+1))(6(\omega+2)n^3-(9\omega+19)n^2-3(\omega+1)(3\omega+1)n+9(\omega+1)^2)}. \quad (108)$$

Let us now focus on the case of dust ($w=0$). The above solution becomes

$$\Delta_m = K_1 t^{-1} + K_2 t^{\alpha_+|_{w=0}} + K_3 t^{\alpha_-|_{w=0}} - K_4 \frac{C_0}{a_0^2} t^{2-\frac{4n}{3}}, \quad (109)$$

where

$$\alpha_{\pm}|_{w=0} = -\frac{1}{2} \pm \frac{\sqrt{(n-1)(n(32n(8n-19)+417)-81)}}{6(n-1)}, \quad (110)$$

$$K_4|_{w=0} = \frac{9(n(12n-31)+18)}{8(4n-9)(12n^3-19n^2-3n+9)}. \quad (111)$$

A graphical representation of the behavior of the exponent of the modes in (109) as n changes is given in Figure 1. This solution has many interesting features. For $0.33 < n < 0.71$ and $1 < n < 1.32$ [38] the modes $t^{\alpha_{\pm}|_{w=0}}$ become

oscillatory. However since the real part of the exponents $\alpha_{\pm}|_{w=0}$ is always negative the oscillation are damped and bound to become subdominant at late times. The appearance of this kind of modes is not associated with any peculiar behavior of the thermodynamic quantities in the background i.e. none of the energy condition are violated for the values of n which are associated with the oscillations. The nature of these oscillations is then an higher order phenomenon. Here we will not undertake a detailed investigation of the origin of these modes, such a study will be left for a future work. Also, for most of the values of n the perturbations grow faster in R^n -gravity than in GR. In fact only for $1.32 \leq n < 1.43$ all the modes grow with a rate slower than $t^{2/3}$.

Probably the most striking feature of the solutions (109) and (106) is that the long wavelength perturbations grow for every value of n , even if the universe is in a state of accelerated expansion (see Figure 1). This is somehow expected from the fact that in [8] the fixed point representing our background is unstable for every value of the parameters. However, the consequence of this feature is quite impressive because it implies that in R^n gravity large scale structures can in principle also be formed in accelerating backgrounds. This is not possible in General Relativity, where it is well known that as soon as the deceleration parameter becomes positive the modes of the Δ solutions (or density contrast) are both decreasing. The suppression of perturbations due to the presence of classical forms of Dark Energy (DE) is one of the most important sources of constraints on the nature of DE itself. Our example shows that if one considers DE as a manifestation of the non-Einsteinian nature of the gravitational interaction on large scales, there is the possibility to have an accelerated expanding background that is compatible with the growth of structures. Of course, in order to better understand this effect, one should also analyze the evolution of perturbations on small scales. However this analysis is beyond the scope of this paper and it is left to a future, more detailed investigation.

In the limit $n \rightarrow 1$ two of the modes of (106) reproduce the two classical modes $t^{2/3}$ and t^{-1} typical of GR, but the other two diverge. At first glance this might be surprising but it does not represent a real pathology of the model. In fact equation (102) reduces to a first order differential equation when $n = 1$. Therefore in this case the two modes in the solution can be discarded and GR is recovered.

From the system (99) we can also obtain the solution for the other scalars:

$$\mathcal{R} = K_5 t^{\frac{2n\omega}{\omega+1}-3} + K_6 t^{\beta+} + K_7 t^{\beta-} - K_8 \frac{C_0}{a_0^2} t^{-\frac{4n}{3(\omega+1)}} \quad (112)$$

$$\mathcal{R} = K_9 t^{\frac{2n\omega}{\omega+1}-1} + K_{10} t^{\gamma+} + K_{11} t^{\gamma-} - K_{12} \frac{C_0}{a_0^2} t^{-\frac{4n}{3(\omega+1)}-1} \quad (113)$$

where

$$\beta_{\pm} = \alpha_{\pm} - 2, \quad (114)$$

$$\gamma_{\pm} = \alpha_{\pm} - 3 \quad (115)$$

and the constants K_5, \dots, K_{12} are all functions of K_1, \dots, K_4 . These expression are rather complicated and will not be given here. It is interesting that these quantities have an oscillatory behavior for the same values of n for which Δ_m is oscillating. Also for these quantities the oscillating modes are always decreasing.

Finally it is useful to derive an expression for the Newtonian potential Φ_N given in (63) which for our background takes the form

$$\begin{aligned} \Phi_N = & \frac{4na_0^2 K_1 t^{\frac{2n\omega}{\omega+1} + \frac{4n}{3(\omega+1)} - 3}}{3(\omega+1)^2} + \frac{4n(2n\omega - (\omega+1)\alpha_-) a_0^2 K_2 t^{\frac{4n}{3(\omega+1)} + \beta_-}}{3(\omega+1)^3} \\ & + \frac{4n(2n\omega - (\omega+1)\alpha_+) a_0^2 K_3 t^{\frac{4n}{3(\omega+1)} + \beta_+}}{3(\omega+1)^3} + \frac{9(\omega+1)^3 - 16n(2n+3(n-1)\omega-3)K_4}{18(\omega+1)^3} C_0. \end{aligned} \quad (116)$$

As in the GR case, this potential has a constant mode and at least one monotonic mode. The presence of a constant mode it is important because it is consistent with the standard Sachs-Wolfe plateau in GR. In addition, the fourth order correction induce oscillations in two of the modes of (116) as it is expected from (63) and the form of the solution for Δ_m . As for Δ_m these oscillatory modes always decay. Again, when $n \rightarrow 1$ the Newtonian potential reduces to the one obtained in General Relativity.

In the case of dust we have

$$\Phi_N = \frac{4n}{3} a_0^2 K_1 t^{\frac{4n}{3}-3} - \frac{4n}{3} a_0^2 K_2 \alpha_-|_{w=0} t^{\frac{4n}{3} + \beta_-|_{w=0}} - \frac{4n}{3} a_0^2 K_3 \alpha_+|_{w=0} t^{\frac{4n}{3} + \beta_+|_{w=0}} + \left[\frac{1}{2} - \frac{8}{9} n(2n-3)K_4 \right] C_0 \quad (117)$$

(see Figure 2). This potential is weaker than the GR one at early times and became stronger at late times. The non constant modes are decaying only for $-0.63 < n \leq 0.33$, $0.71 \leq n < 0.86$ and $1.32 \leq n < 1.36$ and their exponent is greater than the one in GR for any n but $0.28 < n \leq 0.33$.

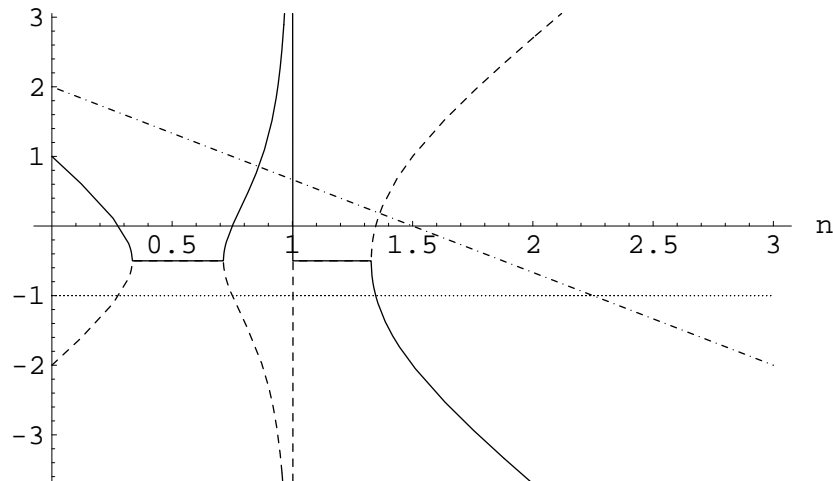


FIG. 1: Plot of the real part of the exponents of each modes of the solution (106) against n in the case of dust ($w = 0$). The continuous and dashed line represent the modes $t^{\alpha_{\pm}}$ respectively (note how they coincide when α_{\pm} are complex), the dashed-dot line represents the mode $t^{2-\frac{4n}{3(w+1)}}$ and the dot line the mode t^{-1} .

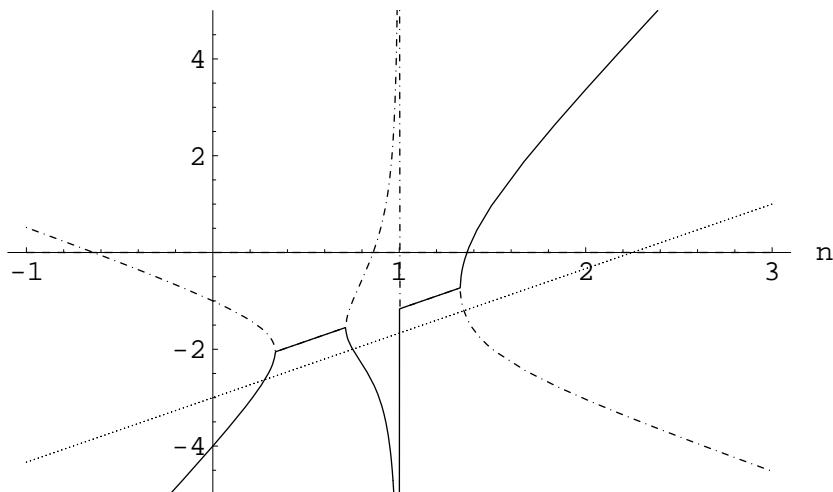


FIG. 2: Plot of the real part of exponents of the modes of the Newtonian Potential (??) against n in the case of dust ($w = 0$). The continuous and dot-dashed line represent the modes $t^{\frac{4n}{3} + \beta_{\pm}|_{w=0}}$ respectively (note how they coincide when β_{\pm} are complex), the dotted line represents the mode $t^{\frac{4n}{3} - 3}$.

VII. CONCLUSION

In this paper we have analyzed in a rigorous and mathematically well defined way the evolution of density perturbations of FLRW backgrounds in fourth order gravity.

Our analysis has been based on two important steps. The first one follows from the fact that in homogeneous and isotropic spacetimes the field equations for a generic fourth order gravity theory can be rewritten in a form that resembles standard GR plus two effective fluids. Then, using 1+3 covariant approach, it is possible to derive the general equations describing the evolution of the cosmological perturbations of these models in FLRW background. In this paper we have dealt only with the evolution of the scalar perturbations, the evolution of tensor and vector perturbations will be presented elsewhere [30, 31].

Providing that one has a clear picture in mind of the effective nature of the fluids involved, the approach above has the advantage of making the treatment of the perturbations physically clear and mathematically rigorous. In particular, it allows one to understand in a natural way that the equations governing the scalar perturbations in

fourth order gravity are of order four rather than two.

Once the general perturbations equations were derived, we applied them to the simple R^n -gravity model. In the long wavelength limit and using a background solution derived from an earlier dynamical systems analysis [8], we were able to find exact solutions to the perturbations equations. The results we obtain are particularly interesting. As expected, our background solution proved to be unstable under scalar perturbations - this solution always corresponds to a saddle point in the phase space of the R^n homogeneous and isotropic models [8]. In addition, for specific intervals of values of the parameter n two of the four modes of the solution can become oscillatory and this might have consequences on the scalar perturbation spectrum. The connection between the spectrum of matter perturbations and the CMB power spectrum then offers an interesting independent way of testing these models on cosmological scales

However the most striking property of the evolution of the density perturbations in this model is that growth is possible even if the background is accelerating. This means that *unlike all the other models for dark energy, in R^n -gravity a decelerated phase is not necessarily required to form the large scale structure*. Of course in order to fully support this claim a detailed analysis of the small-scale perturbations is necessary in order to understand better, if in this regime small scale structure formation is also preserved. However, the idea is very intriguing. A number of important constraints on the nature of Dark Energy come from the requirement that we have a matter dominated phase able to support the formation of large-scale structures. What we have discovered is that in R^n -gravity this is not necessarily the case.

Also the Newtonian potential has a number of interesting features. First of all it contains a constant mode like in GR. This is an encouraging result as this is compatible with the Sachs-Wolfe plateau. On the other hand, the time evolution of the Newtonian potential is essentially different from GR. Such differences suggest that the dynamics of structure formation in R^n -gravity might be very different from what we find in GR and it is definitely worth a more detailed study.

A natural question arises. How general are these results in terms of the form of the fourth order Lagrangian? Or in other words are also other, more popular, fourth order models able to give rise to the same effects? The question is not easy to answer. From the form of the decoupled equation for the matter density perturbations, we can conclude that features like the oscillating modes should be a common to all fourth order theories. On the other hand proving that density perturbations grow for all accelerating backgrounds is a much difficult matter. One interesting hint comes from the general dynamical system analysis given in [9] in which backgrounds similar to the one we used for R^n -gravity are often unstable for most the values of the parameters of that theory. Finally, from the point of view of the dynamics of structure formation, it seems to us reasonable to say that if a model as “close” to GR as R^n -gravity has so many different features, fourth order gravity models with a Lagrangian very different from the one in the Hilbert-Einstein action will in general have a very different dynamics. However, as we have seen in R^n gravity, these differences do not necessarily imply a complete incompatibility with the data coming from the CMB and other observational constraints. However, much more work will have to be done before we can determine whether alternative gravity provides a viable alternative to our much cherished theory of General Relativity.

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APPENDIX A: GENERAL PROPAGATION AND CONSTRAINT EQUATIONS

For a general, imperfect energy-momentum tensor the propagation and constraint equations are:

Expansion propagation (generalized Raychaudhuri equation):

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \sigma_{ab}\sigma^{ab} - 2\omega_a\omega^a - \tilde{\nabla}^a\dot{u}_a + \dot{u}_a\dot{u}^a + \frac{1}{2}(\mu + 3p) = 0$$

Vorticity propagation:

$$\dot{\omega}_{\langle a} + \frac{2}{3}\Theta\omega_a + \frac{1}{2}\text{curl}\dot{u}_a - \sigma_{ab}\omega^b = 0. \quad (\text{A1})$$

Shear propagation:

$$\dot{\sigma}_{\langle ab} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} - \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}^c + \omega_{\langle a}\omega_{b\rangle} - \dot{u}_{\langle a}\dot{u}_{b\rangle} - \frac{1}{2}\pi_{ab} = 0 \quad (\text{A2})$$

Gravito-electric propagation:

$$\begin{aligned} \dot{E}_{\langle ab} + \Theta E_{ab} - \text{curl}H_{ab} + \frac{1}{2}(\mu + p)\sigma_{ab} - 2\dot{u}^c\varepsilon_{cd\langle a}H_{b\rangle}^d - 3\sigma_{c\langle a}E_{b\rangle}^c + \omega^c\varepsilon_{cd\langle a}E_{b\rangle}^d \\ + \frac{1}{2}\dot{\pi}_{\langle ab} + \frac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle} + \frac{1}{6}\Theta\pi_{ab} + \frac{1}{2}\sigma^c{}_{\langle a}\pi_{b\rangle}^c + \frac{1}{2}\omega^c\varepsilon_{c\langle a}\pi_{b\rangle}^d = 0 \end{aligned} \quad (\text{A3})$$

Gravito-magnetic propagation:

$$\begin{aligned} \dot{H}_{\langle ab} + \Theta H_{ab} + \text{curl}E_{ab} - 3\sigma_{c\langle a}H_{b\rangle}^c + \omega^c\varepsilon_{cd\langle a}H_{b\rangle}^d + 2\dot{u}^c\varepsilon_{cd\langle a}E_{b\rangle}^d \\ = \frac{1}{2}\text{curl}\pi_{ab} - \frac{3}{2}\omega_{\langle a}q_{b\rangle} + \frac{1}{2}\sigma^c{}_{\langle a}\varepsilon_{b\rangle}^c q_d \end{aligned} \quad (\text{A4})$$

Vorticity constraint:

$$\tilde{\nabla}^a\omega_a - \dot{u}^a\omega_a = 0. \quad (\text{A5})$$

Shear constraint:

$$\tilde{\nabla}^b\sigma_{ab} - \text{curl}\omega_a - \frac{2}{3}\tilde{\nabla}_a\Theta + 2[\omega, \dot{u}]_a - q_a = 0 \quad (\text{A6})$$

Gravito-magnetic constraint:

$$\text{curl}\sigma_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} + 2\dot{u}_{\langle a}\omega_{b\rangle} = 0. \quad (\text{A7})$$

Gravito-electric divergence:

$$\tilde{\nabla}^b E_{ab} - \frac{1}{3}\tilde{\nabla}_a\tilde{\mu}^m - [\sigma, H]_a + 3H_{ab}\omega^b - \frac{1}{2}\sigma_a^b q_b + \frac{3}{2}[\omega, q]_a + \frac{1}{2}\tilde{\nabla}^b\pi_{ab} - \frac{1}{3}\tilde{\nabla}_a\mu + \frac{1}{3}\Theta q_a = 0 \quad (\text{A8})$$

Gravito-magnetic divergence:

$$\tilde{\nabla}^b H_{ab} - (\mu + p)\omega_a + [\sigma, E]_a - 3E_{ab}\omega^b + \frac{1}{2}\text{curl}q_a + \frac{1}{2}[\sigma, \pi]_a - \frac{1}{2}\pi_{ab}\omega^b = 0 \quad (\text{A9})$$

Here $\omega_a = \frac{1}{2}\varepsilon_a{}^{bc}\omega_{bc}$ and the covariant tensor commutator is

$$[W, Z]_a = \varepsilon_{acd}W^c Z^{de}.$$

APPENDIX B: COVARIANT FORMALISM VERSUS BARDEEN'S FORMALISM

As we have seen the covariant approach is a very useful framework for studying perturbations in alternative theories of gravity. However, since most work on cosmological perturbations is usually done using the Bardeen approach [15], we will give here a brief summary of how can relate our quantities to the standard Bardeen potentials. A detailed analysis of the connection between these formalism is given in [16]. Here we limit ourselves to give the main results for scalar perturbations.

In Bardeen's approach to perturbations of FLRW spacetime, the metric g_{ab} is the fundamental object. If \bar{g}_{ab} is the background metric and $g_{ab} = \bar{g}_{ab} + \delta g_{ab}$ defines the metric perturbations δg_{ab} in these coordinates.

The perturbed metric can be written in the form

$$ds^2 = S^2(\eta)\{- (1 + 2A)d\eta^2 - 2B_\alpha dx^\alpha d\eta + [(1 + 2H_L)\gamma_{\alpha\beta} + 2H_{\alpha\beta}^T]dx^\alpha dx^\beta\}, \quad (\text{B1})$$

where η is the conformal time, and the spatial coordinates are left arbitrary. This spacetime can be foliated in 3-hypersurfaces Σ characterized by constant conformal time η and metric $\gamma_{\alpha\beta}$.

The quantities A and B_α are respectively the perturbation in the lapse function (i.e. the ratio of the proper time distance and the coordinate time one between two constant time hypersurfaces) and in the shift vector (i.e. the rate of deviation of a constant space coordinate line from the normal line to a constant time hypersurface), H_L represents the amplitude of perturbation of a unit spatial volume and $H_{\alpha\beta}^T$ is the amplitude of anisotropic distortion of each constant time hypersurface [32].

The minimal set of perturbation variables is completed by defining the fluctuations in the energy density:

$$\mu = \bar{\mu} + \delta\mu, \quad \delta \equiv \delta\mu/\bar{\mu}, \quad (\text{B2})$$

and the fluid velocity:

$$u^a = \bar{u}^a + \delta u^a, \quad \delta u^\alpha = \bar{u}^0 v^\alpha, \quad \delta u^0 = -\bar{u}^0 A, \quad (\text{B3})$$

together with the energy flux q_a and the anisotropic pressure π_{ab} which are GI by themselves.

These quantities are treated as 3-fields propagating on the background 3-geometry. With suitable choice of boundary conditions [28], these quantities can be uniquely (but non-locally) decomposed into scalars, 3-vectors and 3-tensors:

$$B_\alpha = B|_\alpha + B_\alpha^S, \quad (\text{B4})$$

$$H_{T\alpha\beta} = \nabla_{\alpha\beta} H_T + H_{T(\alpha|\beta)}^S + H_{T\alpha\beta}^{TT}, \quad (\text{B5})$$

where the slash indicates covariant differentiation with respect to the the metric $\gamma_{\alpha\beta}$ of Σ . In this way $\nabla_{\alpha\beta} f = f|_{\beta\alpha} - \frac{1}{3}\nabla^2 f$ and $\nabla^2 f = f|^\gamma|_\gamma$ is the Laplacian. The superscript S on a vector means it is solenoidal ($B_\alpha^{S|\alpha} = 0$), and TT tensors are transverse ($H_{T\alpha}^{TT\beta}|_\beta = 0$) and trace-free.

On the base of (B4) and (B5), it is standard to define *scalar* perturbations as those quantities which are 3-scalars, or are derived from a scalar through linear operations involving only the metric $\gamma_{\alpha\beta}$ and its $|$ derivative. Quantities derived from similar operations on solenoidal vectors and on TT tensors are dubbed *vector* and *tensor* perturbations. Scalar perturbations are relevant to matter clumping, i.e. correspond to density perturbations, while vector and tensor perturbations correspond to rotational perturbations and gravitational waves.

Given the homogeneity and isotropy of the background, we can separate each variable into its time and spatial dependence using the method of harmonic decomposition. In the Bardeen approach the standard harmonic decomposition is performed using the eigenfunctions of the Laplace-Beltrami operator on 3-hypersurfaces of constant curvature Σ (i.e. on the homogeneous spatial sections of FLRW universes). In particular these harmonics are defined by

$$\nabla^2 Y^{(k)} = -k^2 Y^{(k)}, \quad (\text{B6})$$

$$\nabla^2 Y_\alpha^{(k)} = -k^2 Y_\alpha^{(k)}, \quad (\text{B7})$$

$$\nabla^2 Y_{\alpha\beta}^{(k)} = -k^2 Y_{\alpha\beta}^{(k)}, \quad (\text{B8})$$

where $Y^{(k)}$, $Y_\alpha^{(k)}$, $Y_{\alpha\beta}^{(k)}$ are the scalar, vector and tensor harmonics of order k . In this way one can decompose scalars, vectors and tensors as

$$f = f(\eta)Y \quad (\text{B9})$$

$$B_\alpha = B^{(0)}(\eta)Y_\alpha^{(0)} + B^{(1)}(\eta)Y_\alpha^{(1)}, \quad (\text{B10})$$

$$H_{T\alpha\beta} = H_T^{(0)}(\eta)Y_{\alpha\beta}^{(0)} + H_T^{(1)}(\eta)Y_{\alpha\beta}^{(1)} + H_T^{(2)}(\eta)Y_{\alpha\beta}^{(2)}. \quad (\text{B11})$$

The key property of linear perturbation theory of FLRW spacetimes, arising from the unicity of the splitting (B4), (B5), is that in any vector and tensor equation the scalar, vector and tensor parts on each side are separately equal.

All the quantities define above can be decomposed in this way. However, before proceeding, one should note that the quantities A , B_α , H^L , $H_{\alpha\beta}^T$, δ , v^α change their values under a change of correspondence between the perturbed ‘‘world’’ and the unperturbed background, i.e., under a *gauge transformation*. In order to have a gauge-invariant theory one

has to look for combinations of these quantities which are gauge invariant. Bardeen constructed the following GI variables to treat scalar perturbations [15] (giving only their k -space representation):

$$\Phi_A = \left\{ A + \frac{1}{k} \left(B^{(0)'} + \frac{S'}{S} B^{(0)} \right) - \frac{1}{k^2} \left(H_T^{(0)''} + \frac{S'}{S} H_T^{(0)'} \right) \right\} Y, \quad (\text{B12})$$

$$\Phi_H = \left\{ H_L + \frac{1}{3} H_T^{(0)} + \frac{S'}{kS} \left(B^{(0)} - \frac{1}{k} H_T^{(0)'} \right) \right\} Y, \quad (\text{B13})$$

$$V_S = \left(v^{(0)} - \frac{1}{k} H_T^{(0)'} \right) Y_\alpha^{(0)}, \quad (\text{B14})$$

$$\varepsilon_m = \left[\delta(\eta) + 3(1+w) \frac{S'}{kS} (v^{(0)} - B^{(0)}) \right] Y, \quad (\text{B15})$$

where the prime denotes derivative with respect to the conformal time η . Note that there is not a preferred choice of GI density perturbation in this context, as many other GI combinations are possible [32].

The variables covariantly defined in the main text are, by themselves, exact quantities (defined in any spacetime) and are GI by themselves, therefore, to first order, we can express them as linear combinations of Bardeen's GI variables. In [16] this expansions is given in full generality. Here we will limit ourselves to a few examples, giving only the scalar contributions and refer the reader to [16] for details.

The scalar part of the shear, electric part of the Weyl tensor, energy flux and anisotropic pressure are given by

$$\sigma_{\alpha\beta} = -SkV_S^{(0)} Y_{\alpha\beta}^{(0)} \quad (\text{B16})$$

$$E_{\alpha\beta} = \frac{1}{2} k^2 (\Phi_A - \Phi_H) Y_{\alpha\beta}^{(0)} \quad (\text{B17})$$

$$q_\alpha = S \left[\kappa h V_S^{(0)} + \frac{2k}{S^2} \left(\Phi_H' - \frac{S'}{S} \Phi_A \right) \right] Y_\alpha^{(0)} \quad (\text{B18})$$

$$\pi_{\alpha\beta} = -\frac{k^2}{S^2} (\Phi_H + \Phi_A) Y_{\alpha\beta}^{(0)}. \quad (\text{B19})$$

while the scalar parts of the energy density, expansion and 3-curvature scalar gradients can be written as

$$\mathcal{D}_\alpha = -ka \varepsilon_m(\eta) Y_\alpha^{(0)}, \quad (\text{B20})$$

$$\mathcal{Z}_\alpha = \left\{ -3k \left(\Phi_H' - \frac{S'}{S} \Phi_A \right) + [(3K - k^2) - \frac{3}{2} \kappa h S^2] V_S^{(0)} \right\} Y_\alpha^{(0)}, \quad (\text{B21})$$

$$C_\alpha = -4Sk(k^2 - 3K) \left(\Phi_H - \frac{S'}{ka} V_S^{(0)} \right) Y_\alpha^{(0)}. \quad (\text{B22})$$

Finally the key covariant scalar perturbation variables are given by

$$\Delta = S^{(3)} \nabla^a \mathcal{D}_a = -k^2 \varepsilon_m(\eta) Y, \quad (\text{B23})$$

$$C = S^{(3)} \nabla^a C_a = k^2 S^2 R^*(\eta) Y. \quad (\text{B24})$$

The relations above can be used to give an intrinsic physical and geometrical meaning to Bardeen's variables, and also to recover his equations. For example, from the (B16) the variables V_S can be recognized as the scalar contribution to the shear.

The variables ε_m (B15), interpreted by Bardeen as the usual density perturbation $\delta\mu/\mu$ within the comoving gauges $v - B = 0$, acquire a covariant significance as the scalar "potential" for the fractional density gradient \mathcal{D}_a (B21). Obviously, this quantity also represents the potential for the divergence Δ (B23) of \mathcal{D}_a (or its harmonic component).

The two independent GI metric potentials Φ_A and Φ_H can be combined in such a way to give

$$\Phi_\pi = \frac{1}{2} (\Phi_H + \Phi_A), \quad \Phi_N = \frac{1}{2} (\Phi_A - \Phi_H); \quad (\text{B25})$$

the former Φ_π is a stress potential while the latter Φ_N plays exactly the role of a Newtonian gravitational potential. This last interpretation follows directly through (B17), where the scalar part of $E_{\alpha\beta}$ has exactly the same form it has in Newtonian theory $E_{\alpha\beta} = \nabla_{\alpha\beta} \Phi_N$ [26], independently of any gauge choice. For a perfect fluid ($\pi_{ab} = 0$) Φ_A and Φ_H are proportional to each other i.e. $\Phi_A = -\Phi_H$, however in our case, the "curvature" fluid is imperfect, so this is not the case. In fact, we can see from equation (24) that the anisotropic pressure π_{ab} is related to the shear of the matter

flow and therefore the scalar potentials Φ_A and Φ_H are related to the Bardeen shear variable $V_S^{(0)}$. This relationship can be calculated explicitly using (24) together with equations (B16), (B19) and the expression for the Ricci scalar $R = 2 \left(\dot{\Theta} + \frac{2}{3}\Theta^2 + \frac{1}{2}\tilde{R} \right)$. After a lengthy calculation we obtain

$$\begin{aligned} \Phi_A + \Phi_H &= \frac{f''}{f'} \left\{ 6\Phi_H'' + 8\frac{S'}{S}\Phi_H' + 4(k^2 - 3K)\Phi_H + 2\frac{S'}{S}(\Phi_A' + 3\frac{S'}{S}\Phi_A) \right. \\ &\quad \left. + 2k^{-1} \left[(3K - k^2) - \frac{3}{2}ha^2 \right] V_S' + k^{-1} \left[\frac{4}{3}\frac{S'}{S}(k^2 - 3K) - 8S'ah - 3S^2h' \right] V_S \right\}. \end{aligned} \quad (\text{B26})$$

APPENDIX C: COVARIANT IDENTITIES

On a flat Friedmann background, the following covariant linearized identities hold:

$$\tilde{\nabla}_a \dot{f} = (\tilde{\nabla}_a f)' + \frac{1}{3}\Theta \tilde{\nabla}_a f - \dot{f} \dot{u}_a, \quad (\text{C1})$$

$$\tilde{\nabla}^2 (\tilde{\nabla}_a f) = \tilde{\nabla}_a (\tilde{\nabla}^2 f) - \frac{2K}{S^2} \tilde{\nabla}_a f + 2\dot{f} \omega_a, \quad (\text{C2})$$

$$\tilde{\nabla}^2 \dot{f} = (\tilde{\nabla}^2 f)' + \frac{2}{3}\Theta \tilde{\nabla}^2 f - \dot{f} \tilde{\nabla}^a \dot{u}_a, \quad (\text{C3})$$

$$(\tilde{\nabla}_a V_b)' = \tilde{\nabla}_a \dot{V}_b - \frac{1}{3}\Theta \tilde{\nabla}_a V_b, \quad (\text{C4})$$

$$\tilde{\nabla}_{[a} \tilde{\nabla}_{b]} V_c = -\frac{K}{S^2} V_{[a} h_{b]c}, \quad (\text{C5})$$

$$\tilde{\nabla}^b \tilde{\nabla}_{\langle a} V_{b \rangle} = \frac{1}{2} \tilde{\nabla}^2 V_a + \frac{1}{6} \tilde{\nabla}_a (\tilde{\nabla}^b V_b) + \frac{K}{S^2} V_a, \quad (\text{C6})$$

$$(\tilde{\nabla}_a W_{cd})' = \tilde{\nabla}_a \dot{W}_{cd} - \frac{1}{3}\Theta \tilde{\nabla}_a W_{cd}, \quad (\text{C7})$$

where $V_a = V_{\langle a}$ and $W_{ab} = W_{\langle ab}$ are first order quantities.

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- [34] Another possible choice of frame would be $u_a^R = \nabla_a R$ in analogy with the scalar field case [21, 27]. However, this choice would imply that in the 3-spaces orthogonal to u_a^R , the projected spatial gradients of R vanish, putting a further condition on the structure of the perturbed spacetime which may not be satisfied in general.
- [35] Nevertheless, our formalism can be easily connected with the Bardeen or other metric approaches. See Appendix B for details.
- [36] This last assumption does not compromise the generality of our treatment because it is well known that although the 3-gradient of the vorticity acts as a source term for the evolution equation of \mathcal{D}_a it does not affect the scalar part of this quantity [18].
- [37] Note that the underlying assumption in this decomposition is that the perturbation variables can be factorized into purely temporal and purely spatial components.
- [38] The values of n presented here and in the following are the result of the resolution of algebraic equations of order greater than two. These values have been necessarily calculated numerically. This means that the values we will give for the interval n will be necessarily an approximation.